Europ. J. Combinatorics (1986) 7, 171–176

Adjacency in Binary Matroids

P. D. SEYMOUR*

We say that two elements e, f of a binary matroid M are 'adjacent' if there is no minor of M isomorphic to $\mathcal{M}(K_4)$ which uses both e and f and in which they correspond to opposite edges. We give a good characterization of when two elements are adjacent. In particular, we show that if M is 4-connected, elements e, f are adjacent if and only if M is either graphic or cographic and the elements correspond to adjacent edges of the graph. We deduce a theorem about disjoint paths in graphs.

1. INTRODUCTION

We shall assume familiarity with matroid theory—for an introduction, see Welsh [6]. Throughout this paper we shall be concerned only with binary matroids. Let e, f be distinct elements of a binary matroid M. We say that e, f are *adjacent* in M if M has no minor N with the following properties:

(M1) $e, f \in E(N)$

(M2) $N \cong \mathcal{M}(K_4)$

(M3) *no circuit of N of cardinality 3 contains both e and f (that is, e and f correspond to non-adjacent edges of K_4).*

[Some notation: $E(M)$ is the element set of M; K_4 is the complete graph with four vertices; $\mathcal{M}(G)$ denotes the polygon matroid of the graph G, and $\mathcal{M}^*(G)$ its bond matroid; $\cong$ denotes isomorphism.]

This definition represents an attempt to find an extension to binary matroids of the adjacency relation in graphs (of *edges*—two edges are adjacent in a graph if they have a common end). It is motivated by the following observation.

1.1 *If e, f are adjacent edges of a graph G then they are adjacent in any graph obtained from G by deleting and contracting edges different from e, f, and in particular they are adjacent in $\mathcal{M}(G)$.*

We also evidently have

1.2 *If e, f are adjacent in M then they are adjacent in M^*.*

[M^* denotes the dual of M.]

These observations imply that if e, f are adjacent edges of a graph G then they are adjacent in both $\mathcal{M}(G)$ and $\mathcal{M}^*(G)$. Our main result is a characterization when elements e, f are adjacent in a binary matroid. We find in particular a partial converse to the foregoing; if M is 4-connected (defined later), and e, f are adjacent in M then there is a graph G in which e, f are adjacent edges such that $M = \mathcal{M}(G)$ or $\mathcal{M}^*(G)$. If we specialize this result to graphic matroids M, we obtain the following.

1.3 *If e, f are adjacent elements of $\mathcal{M}(G)$ and $\mathcal{M}(G)$ is 4-connected, then either e, f are adjacent edges of G or G is planar and can be drawn in the plane with e, f on a common region.*

* Partially supported by National Research Foundation grant number MCS-8103440.

0195-6698/86/020171+06 $02.00/0

This is a disguised version of an earlier theorem of the author concerning disjoint paths in graphs [3], and the derivation here provides an independent proof, described in section 4. Section 2 contains some preliminaries, and the main result is proved in section 3.

2. Connectivity and k-Sums

If M is a matroid and $k>0$ is an integer, a partition (X_1, X_2) of $E(M)$ is called a *k-separation* of M if $|X_1|, |X_2| \geq k$ and

$$r_M(X_1)+r_M(X_2) \leq r_M(E(M))+k-1.$$

$r_M(X)$ denotes the rank in M of $X \subseteq E(M)$. M is said to be *k-connected* if it has no k'-separation with $k'<k$. 'Connected' means '2-connected'.

Let M_1, M_2 be binary matroids with elements sets E_1, E_2, where E_1, E_2 may intersect. We define $M_1 \triangle M_2$ to be the binary matroid with element set $E_1 \triangle E_2$ and with cycles all subsets of $E_1 \triangle E_2$ of the form $C_1 \triangle C_2$, where C_i is a cycle of M_i $(i=1,2)$. [For sets X_1, X_2, $X_1 \triangle X_2$ denotes $(X_1-X_2)\cup(X_2-X_1)$. A *cycle* of a binary matroid is a subset of the elements expressible as a disjoint union of circuits.]

We are only concerned with two special cases of this operation, as follows:

(a) When $|E_1 \cap E_2|=1$, and $E_1 \cap E_2=\{z\}$ say, and z is not a loop or coloop of M_1 or M_2, and $|E_1|$, $|E_2| \geq 3$, $M_1 \triangle M_2$ is a *2-sum* of M_1 and M_2.

(b) When $|E_1 \cap E_2|=3$, and $E_1 \cap E_2=Z$ say, and Z is a circuit of both M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and $|E_1|$, $|E_2| \geq 7$, $M_1 \triangle M_2$ is a *3-sum* of M_1 and M_2.

In either case M_1 and M_2 are called the *parts* of the k-sum. The following results are proved in [2].

2.1 *Let M be a connected binary matroid and let (X_1, X_2) be a partition of M with $|X_1|$, $|X_2| \geq 2$. Then there are binary matroids M_1, M_2 with $E(M_1)-E(M_2)=X_1$, $E(M_2)-E(M_1)=X_2$, such that M is the 2-sum of M_1 and M_2, if and only if (X_1, X_2) is a 2-separation of M. If so, then M_1, M_2 are isomorphic to minors of M.*

2.2 *Let M be a 3-connected binary matroid and let (X_1, X_2) be a partition of M with $|X_1|$, $|X_2| \geq 4$. Then there are binary matroids M_1, M_2 with $E(M_1)-E(M_2)=X_1$, $E(M_2)-E(M_1)=X_2$, such that M is the 3-sum of M_1 and M_2, if and only if (X_1, X_2) is a 3-separation of M. If so, then M_1, M_2 are isomorphic to minors of M.*

We shall also require the following results from [1, 2].

2.3 *For $k=2$ or 3, if M is k-connected and is the k-sum of M_1, M_2 then M is regular if and only if M_1, M_2 are both regular.*

2.4 *If M is 3-connected and regular, then either M has a 3-separation (X_1, X_2) with $|X_1|, |X_2| \geq 6$, or M is graphic, or M is cographic, or M is isomorphic to R_{10}.*

[R_{10} is a particular matroid with ten elements. Here the only fact about R_{10} we need is that it has no circuit of cardinality 3.]

3. The Characterization

We now give our main result. It is most conveniently presented by means of a series of alternatives, as follows. [If $X \subseteq E(M)$, $M\backslash X$ denotes the matroid with element set $E(M)-X$ and the induced independence structure, and M/X denotes $(M^*\backslash X)^*$.]

3.1 *Let e, f be distinct elements of a binary matroid M.*
(a) *If M has a 1-separation (X_1, X_2) with $e, f \in X_1$, then e, f are adjacent in M if and only if they are adjacent in $M \backslash X_2$.*
(b) *If M has a 1-separation (X_1, X_2) with $e \in X_1$, $f \in X_2$, then e, f are adjacent in M.*
(c) *If M is connected and has a 2-separation (X_1, X_2) with $e, f \in X_1$, let M_1, M_2 be the parts of the corresponding 2-sum; then e, f are adjacent in M if and only if they are adjacent in M_1.*
(d) *If M is connected and has a 2-separation (X_1, X_2) with $e \in X_1$, $f \in X_2$, let M_1, M_2 be the parts of the corresponding 2-sum; then e, f are adjacent in M if and only if e, z are adjacent in M_1 and z, f are adjacent in M_2, where $E(M_1) \cap E(M_2) = \{z\}$.*
(e) *If M is 3-connected and not regular, then e, f are not adjacent in M.*
(f) *If M is 3-connected and regular, and has a 3-separation (X_1, X_2) with $e, f \in X_1$ and $|X_1|, |X_2| \geq 4$, let M_1, M_2 be the parts of the corresponding 3-sum; then e, f are adjacent in M if and only if they are adjacent in M_1.*
(g) *If M is 3-connected and regular, and has no 3-separation (X_1, X_2) with $e, f \in X_1$ and $|X_1|, |X_2| \geq 4$, then e, f are adjacent in M if and only if there is a graph G with $M = \mathcal{M}(G)$ or $\mathcal{M}^*(G)$ such that e, f are adjacent edges of G.*

PROOF. Cases (a)–(d) are easy, since $\mathcal{M}(K_4)$ is 3-connected, and are left to the reader. Case (e) is a theorem of [4]. For cases (f) and (g) we shall need the following lemma.

3.2 *If $\{e, f, g\}$ is a cycle of a binary matroid M, and $M \backslash g$ is regular, then the following are equivalent:*
(a) *M is regular*
(b) *e, f are adjacent in M*
(c) *e, f are adjacent in $M \backslash g$.*

[$M \backslash g$ abbreviates $M \backslash \{g\}$.]

PROOF. (a)$\Rightarrow$(c). If e, f are not adjacent in $M \backslash g$, choose $X, Y \subseteq E(M \backslash g)$ such that $(M \backslash g) \backslash X / Y$ satisfies (M1), (M2), (M3). Put $N = M \backslash X / Y$. Then $\{e, f, g\}$ is a cycle of N and so N is isomorphic to the Fano matroid. Thus M is not regular, by Tutte's theorem [5].

(c)$\Rightarrow$(b). If e, f are not adjacent in M, choose $X, Y \subseteq E(M)$ such that $M \backslash X / Y$ satisfies (M1), (M2), (M3). Now $g \notin Y$, because $\{e, f\}$ is not a cycle of $M \backslash X / Y$, and $g \in X \cup Y$, because $\{e, f, g\}$ is not a cycle of $M \backslash X / Y$, by (M3). Thus $g \in X$, and $M \backslash X / Y$ is a minor of $M \backslash G$. Hence e, f are not adjacent in $M \backslash g$.

(b)$\Rightarrow$(a). We proceed by induction on $|E(M)|$. Suppose then that the result is true for all smaller matroids, and that M is not regular. If M has a 1-separation (X_1, X_2) with $\{e, f, g\} \subseteq X_1$, then $M \backslash X_1$ is regular (since $M \backslash g$ is regular) and so $M \backslash X_2$ is not regular. But $(M \backslash X_2) \backslash g$ is regular, and so by induction e, f are not adjacent in $(M \backslash X_2)$. Hence they are not adjacent in M.

We may assume then that M has no such 1-separation. Now $M \backslash g$ is regular and M is not, and so g is not a loop or parallel element of M. Thus $\{e, f, g\}$ is a circuit of M, and so e, f, g lie in the same component of M. Hence M is connected. If two elements of M are parallel, x, y say, then $x, y \neq g$ (because g is not a parallel element) and $\{x, y\} \neq \{e, f\}$ (since $\{e, f\}$ is a circuit). Thus one of x, y (x say), is distinct from e, f, g. Hence $M \backslash x$ is not regular, and so e, f are not adjacent in $M \backslash x$ by induction, and hence they are not adjacent in M as required. We assume then that M has no parallel elements. If (X_1, X_2) is a 2-separation of M, then without loss of generality we may assume that $|X_1 \cap \{e, f, g\}| \geq 2$. If $X_2 \cap \{e, f, g\} = \{z\}$ say, then z is spanned in M by X_1, and so either

$|X_2-\{z\}|\leq 1$ or $(X_1\cup\{z\}, X_2-\{z\})$ is a 2-separation of M. The first implies that $|X_2|=2$, so that X_2 is a circuit or cocircuit of M. But $|X_2\cap\{e,f,g\}|=1$ and M has no parallel elements, a contradiction. Thus $(X_1\cup\{z\}, X_2-\{z\})$ is a 2-separation of M. We deduce that either M is 3-connected or it has a 2-separation (X_1, X_2) with $e,f,g\in X_1$.

We assume the second. Let M_1, M_2 be the parts of the corresponding 2-sum. Since $\{e,f,g\}$ is a circuit of M, there is a circuit of M intersecting both X_1 and X_2 and not containing g, and so M_2 is isomorphic to a minor of $M\backslash g$ (by a result of [2]). Thus M_2 is regular, and so by 2.3, M_1 is not regular. But $M_1\backslash g$ is regular, since it is isomorphic to a minor of $M\backslash g$, and so by induction e, f are not adjacent in M_1 and hence not in M.

Thus we may assume that M is 3-connected. By case (e) above, there exist $X, Y\subseteq E(M)$ such that $M\backslash X/Y$ satisfies (M1), (M2), (M3), and so (a) holds as required. This completes the proof of the Lemma 3.2.

We now return to the proof of 3.1.

Case (f) Suppose that M is 3-connected and regular, and has a 3-separation (X_1, X_2) with $e,f\in X_1$ and $|X_1|, |X_2|\geq 4$. Let M_1, M_2 be the parts of the corresponding 3-sum. Then M_1, M_2 are regular by 2.3. If for some $g\in E(M)-\{e,f\}$, $\{e,f,g\}$ is a cycle of M, then the same is true for M_1, and by 3.2 e, f are adjacent in both M and M_1. We assume then that there is no such element g.

Take a new element g and let M^+, M_1^+ be the binary matroids in which $\{e,f,g\}$ is a cycle and $M^+\backslash g=M$, $M_1^+\backslash g=M_1$. Then $M^+=M_1^+\triangle M_2$, and M^+ is 3-connected, as is easily seen. So M^+ is regular if and only if M_1^+ is regular, by 2.3. Thus we have

$$\begin{aligned} &e, f \text{ are adjacent in } M\\ &\quad\Leftrightarrow\ M^+ \text{ is regular by (3.2)}\\ &\quad\Leftrightarrow\ M_1^+ \text{ is regular}\\ &\quad\Leftrightarrow\ e, f \text{ are adjacent in } M_1 \text{ by 3.2.}\end{aligned}$$

This proves case (f).

Case (g) The 'if' part follows from 1.1. We must prove 'only if'. Suppose then that M is 3-connected and regular, and has no 3-separation (X_1, X_2) with $e,f\in X_1$ and $|X_1|, |X_2|\geq 4$, and that e, f are adjacent in M. Let g, h be two new elements, and let M^+ be the binary matroid with element set $E(M)\cup\{g,h\}$ such that $M^+\backslash g/h=M$ and $\{e,f,g\}$ is a circuit of M^+ and $\{e,f,h\}$ is a cocircuit of M^+. (It is easy to see that M^+ exists and is uniquely defined by these requirements.)

Now $(M^+/h)\backslash g=M$, and $\{e,f,g\}$ is a circuit of M^+/h, and e, f are adjacent in M. Thus by 3.2 M^+/h is regular, and e, f are adjacent in M^+/h. But $\{e,f,h\}$ is a circuit of $(M^+)^*$, and $(M^+)^*\backslash h$ is regular, and e, f are adjacent in $(M^+)^*\backslash h$; thus by 3.2, $(M^+)^*$ is regular and e, f are adjacent in $(M^+)^*$. Hence M^+ is regular and e, f are adjacent in M^+.

Now we apply 2.4 to M^+. If M^+ has a 3-separation (X_1, X_2) with $|X_1|, |X_2|\geq 6$ then $(X_1-\{g,h\}, X_2-\{g,h\})$ is a 3-separation of M with $|X_1-\{g,h\}|, |X_2-\{g,h\}|\geq 4$, and so by hypothesis we have $e\in X_1$, $f\in X_2$ or $e\in X_2$, $f\in X_1$. Without loss of generality we assume that $e\in X_1, f\in X_2$. By symmetry we may assume that $g\in X_1$; but then X_1 spans f in M^+, and so $(X_1\cup\{f\}, X_2-\{f\})$ is a 3-separation of M^+, and

$$((X_1\cup\{f\})-\{g,h\}, X_2-\{f,h\})$$

is a 3-separation of M. But $\{e,f\}\subseteq(X_1\cup\{f\})-\{g,h\}$, and

$$|(X_1\cup\{f\}-\{g,h\}|\geq 4$$

and

$$|X_2 - \{f, h\}| \geq 4,$$

contrary to our hypothesis about M. Thus M^+ has no such 3-separation. It has a 3-element circuit, and so is not isomorphic to R_{10}; and hence by 2.4, it is either graphic or cographic. Let G^+ be a graph such that $M^+ = \mathcal{M}(G^+)$ or $\mathcal{M}^*(G^+)$. One of $\{e, f, g\}$, $\{e, f, h\}$ is the edge-set of a circuit of G^+, and so e, f are adjacent edges of G^+. The result follows from 1.1.

4. An Application to Graphs

In this section we give a new proof of a graph-theoretic result proved in [3]. We begin by specializing 3.1 to graphic matroids. [In a graph G, $\Delta(A)$ denotes the set of vertices of $V(G) - A$ adjacent to vertices in $A \subseteq V(G)$.]

4.1 *Let e, f be distinct edges of a graph G. Then e, f are adjacent in $\mathcal{M}(G)$ if and only if there are disjoint subsets $A_1, \cdots, A_k \subseteq V(G)$, containing no ends of e, f, such that*
(a) *for $i \neq j$, $\Delta(A_i) \cap A_j = \varnothing$,*
(b) *for $1 \leq i \leq k$, $|\Delta(A_i)| \leq 3$,*
(c) *for $1 \leq i \leq k$, the subgraph of G induced by A_i is connected,*
(d) *if G' is the graph obtained from G by deleting $A_1 \cdots A_k$ and (for each i) adding new edges joining every pair of distinct vertices in $\Delta(A_i)$, then G' is planar and may be drawn in the plane with e, f both on the boundary of the infinite region.*

4.1 follows easily from 3.1 by induction on the size of G, using standard results about polygon matroids of graphs, and we omit the details. The following lemma relates our 'adjacenty' problem to a problem about disjoint paths.

4.2 *Let e, f be disjoint edges of a graph G. Let the ends of e be s_1, s_2 and let the ends of f be t_1, t_2. Then the following are equivalent:*
(a) *e, f are not adjacent in $\mathcal{M}(G)$,*
(b) *there are four paths $P_{11}, P_{12}, P_{21}, P_{22}$ of G, such that P_{11}, P_{22} have no common vertices, P_{12}, P_{21} have no common vertices, and P_{ij} joins s_i to $t_j (i, j = 1, 2)$.*

Proof. (a)$\Rightarrow$(b). Let N be a minor of $\mathcal{M}(G)$ satisfying (M1), (M2), (M3). Let C_1, C_2 be the two circuits of N which contain both e and f. Choose $X, Y \subseteq E(G)$ such that $N = \mathcal{M}(G) \backslash X / Y$, and let C_1', C_2' be circuits of $\mathcal{M}(G)$ such that $C_i' - Y = C_i (i = 1, 2)$. Now for $i = 1, 2$, C_i' corresponds to a circuit of G, D_i say. If we remove e and f from D_1 and D_2 we obtain the required four paths of G.

(b)$\Rightarrow$(a). We proceed by induction on $|E(G)|$. Let g_1 be the edge of P_{12} incident with s_1, and let g_2 be the edge of P_{21} incident with s_2. Let v_i be the end of g_i different from $s_i (i = 1, 2)$. If v_1 is not in P_{22}, the result follows from our inductive hypothesis applied to the graph obtained from G by contracting g_1. We assume then that v_1 is in P_{22}, and, similarly, that v_2 is in P_{11}. But now the existence of the required K_4 minor is clear.

Now let s_1, t_1, s_2, t_2 be distinct vertices of a connected graph G. Consider the following statement P.

P *There are paths of G from s_1 to t_1 and from s_2 to t_2 with no common vertices.*

A characterization of the graphs with statement P false was stated in [3] without proof. In this section we derive the same characterization from (4.1), using (4.2).

Let G_1 be the graph obtained from G by adding two extra edges e_1, f_1 joining s_1 to s_2 and t_1 to t_2, respectively.

4.3 *Statement P is false if and only if the following conditions all hold*:
(a) e_1, f_1 *are adjacent in* $\mathcal{M}(G_1)$,
(b) e_2, f_2 *are adjacent in* $\mathcal{M}(G_2)$,
(c) *if there exist two vertex-disjoint paths from* $\{s_1, s_2\}$ *to* $\{t_1, t_2\}$, *then there exist two vertex-disjoint paths from* $\{s_1, t_1\}$ *to* $\{s_2, t_2\}$.

PROOF. First we prove 'only if'. Suppose then that statement P is false. By 4.2, (a) and (b) hold. To see (c), suppose that there are two vertex-disjoint paths P_1, P_2 from $\{s_1, s_2\}$ to $\{t_1, t_2\}$. We assume without loss of generality that s_i is the initial vertex of P_i $(i = 1, 2)$. Then since Statement P is false, P_1 joins s_1 to t_2, and P_2 joins s_2 to t_1. Hence (c) holds.

To prove 'if', we suppose that statement P is true, and we must show that one of (a)-(c) is false. If (c) is true, then there are vertex-disjoint paths P_1, P_2 from $\{s_1, t_1\}$ to $\{s_2, t_2\}$. Without loss of generality we assume that s_1 is the initial vertex of P_1, and t_1 is the initial vertex of P_2. If P_1 joins s_1 to s_2 and P_2 joins t_1 to t_2 then (b) is false by 4.2. If P_1 joins s_1 to t_2 and P_2 joins t_1 to s_2 then (a) is false by 4.2. This completes the proof.

From 4.1 and 4.3 we can derive the characterization of [3], the following.

4.4 *Statement P is false if and only if there exists disjoint subsets* $A_1, \ldots, A_k \subseteq V(G)$, *containing none of* s_1, t_1, s_2, t_2, *such that*
(a) *for* $i \neq j$, $\Delta(A_i) \cap A_j = \varnothing$,
(b) *for* $1 \leq i \leq k$, $|\Delta(A_i)| = 3$,
(c) *if* G' *is the graph obtained from* G *by (for each* i*) deleting* A_i *and adding new edges joining every pair of distinct vertices in* $\Delta(A_i)$, *then* G' *is planar and may be drawn in the plane with* s_1, s_2, t_1, t_2 *appearing on the boundary of the infinite region in that order.*

PROOF. The 'if' part of (4.4) is easily seen directly. We prove the 'only if' part. Suppose then that statement P is false. By 4.1 and 4.3(a), there exist disjoint subsets $A_1, \ldots, A_k$ satisfying (a), (b), (c), except possibly that the order of appearance of s_1, s_2, t_1, t_2 on the infinite region is s_1, s_2, t_2, t_1; and moreover, the subgraph of G induced by each A_i is connected. If this is the order, and there is a cut-vertex of G' separating s_1, s_2 and t_1, t_2 then G' may be redrawn in the required manner. If there is no such cut-vertex, then there are vertex-disjoint paths of G', P_1, P_2 say, such that P_i joins s_i and t_i $(i = 1, 2)$. But this contradicts the falsity of statement P, as is easily seen by using the fact that each A_i gives a connected subgraph of G.

REFERENCES

1. T. Brylawski, Modular constructions for combinatorial geometries, *Trans. Amer. Math. Soc.* **203** (1975), 1-44.
2. P. D. Seymour, Decomposition of regular matroids, *J. Combinatorial Theory Ser. B* **28** (1980), 305-359.
3. P. D. Seymour, Disjoint paths in graphs, *Discrete Math.* **29** (1980), 293-309.
4. P. D. Seymour, Minors of 3-connected matroids, *Europ. J. Combinatorics*, **6** (1985), 375-382.
5. W. T. Tutte, A homotopy theorem for matroids, I, II, *Trans. Amer. Math. Soc.* **88** (1958), 144-178.
6. D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

Received 8 September 1982

P. D. SEYMOUR
Department of Mathematics, The Ohio State University, 231 W. 18th Avenue, Columbus OH 43210, U.S.A.

Current address: Bell Communications Research, 435 South Street, Morristown, NJ 07960, U.S.A.